Linear CDMA detection algorithm based on statistical neurodynamics and belief propagation and the stability conditions

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Recently a modified algorithm of code-division multiple-access (CDMA) parallel interference canceler (PIC) has been proposed by Tanaka based on statistical neurodynamics. In this paper we apply the modified algorithm to the linear PIC (LPIC) and investigate its stability. We show that the stable (unstable) fixed points of the modified algorithm corresponds to the stable (unstable) replica symmetry solutions with the Gaussian prior. We also show the modified algorithm is a special case of the Kabashima's belief-propagation algorithm with Gaussian prior.

§1. Introduction

We consider the fully-synchronous randomly spread K-user BPSK(two valued)-CDMA channel, with the perfect power control, subject to the additive white Gaussian noise, in which the received signal y^μ at chip interval μ is given by

$$y^\mu = \frac{1}{\sqrt{N}} \sum_{k=1}^{K} s_k^\mu b_k + \sigma_0 n^\mu \quad (\mu = 1, \cdots, N), \tag{1·1}$$

where $b_k \in \{-1, 1\}$ and $\{s_k^\mu\}$ are the BPSK (two valued) information symbol and the spreading code of user k, respectively. The variance of the channel noise is σ_0^2 and $n^\mu \sim N(0, 1)$. The matched filter output h_k, which is a naive single user detection, takes an inner product between the received signal and the spreading code of desirable user k, as

$$h_k = \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} s_k^\mu y^\mu = b_k + \sum_{l \neq k} W_{kl} b_l + \sigma_0 \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} s_k^\mu n^\mu \tag{1·2}$$

where $W_{kl} = \frac{1}{N} \sum_{\mu=1}^{N} s_k^\mu s_l^\mu (1 - \delta_{kl})$ is the off-diagonal correlation matrix of spreading codes. The matched filter output wastes information of the multiple-access interference (MAI) regarding it as a noise. PIC makes use of the information of MAI by iteratively estimating MAI and subtracting the estimated MAI from matched filter output as follows:[1)]

$$\boldsymbol{x}^t = f(\boldsymbol{h} - W \boldsymbol{x}^{t-1}) \tag{1·3}$$

where $\boldsymbol{x}^t = (x_1^t, \cdots, x_K^t)$ is a soft decisions at stage t, $\boldsymbol{h} = (h_1, \cdots, h_K)$, and $f(\cdot)$ is an odd decision function. We take the matched filter initial stage: $\boldsymbol{x}^0 = f(\boldsymbol{h})$. In this paper we study the detectors using a linear tentative decision funtion $f(x) = ax$, which is called linear PIC (LPIC). In the case we make the hard decision to obtain the estimation of stage t as $\hat{b}_k^t = \mathrm{sgn}[x_k^t]$. On the other hand, by calculating inverse matrix of the correlation matrix W we obtain the one shot linear detector with receiver-estimated channel noise σ as follows.

$$\boldsymbol{x} = [W + (1+\sigma^2)I]^{-1}\boldsymbol{h} \tag{1·4}$$

The case neglecting the channel noise $\sigma = 0$ is known as the decorrelating detector and the case that receiver knows the exact channel noise $\sigma = \sigma_0$ is known as the linear MMSE detector.[2] There is a simple relationship between LPIC and the one shot linear detectors, *i.e.* LPICs converge to the one shot linear detectors, if they are convergent. The destination of LPIC is controlled by the system parameter a.

§2. Modified algorithm of Tanaka: PQΓ-algorithm

Tanaka[3] proposed a modified algorithm of PIC on the basis of statistical neurodynamics. Applying the modified algorithm of Tanaka to LPIC we obtain the following algorithm.

$$\boldsymbol{x}^t = a\tilde{\boldsymbol{u}}^{t-1}, \quad \tilde{\boldsymbol{u}}^t = P_t\left(\boldsymbol{h} - W\boldsymbol{x}^t\right) + (1-P_t)\tilde{\boldsymbol{u}}^{t-1} - \boldsymbol{\Gamma}^t \tag{2·1}$$

where the term P_t is imposed to discount unreliable estimated MAI so as to control errors due to subtracting unreliable MAI. This scheme is the partial interference cancellation proposed by Divsalar *et al.*[4] and the term Γ_k^t corresponds to the dynamical analogue of the Onsager reaction term used in neural network theory. The initial conditions are taken as $\boldsymbol{x}^{-1} = \boldsymbol{0},\ P_{-1} = 1,\ \boldsymbol{\Gamma}^{-1} = \boldsymbol{0}$. Tanaka introduced the partial cancellation factor P_t so as to eliminate interstage correlation of decision statistics as $\langle \tilde{u}_k^t \tilde{u}_k^t \rangle = \delta_{ts} C_t$ and subtracts the dynamical Onsager term so as to make the averaged (macroscopic) dynamics, which obtained with the Gaussian assumption for the decision statistics $\tilde{u}_k^t$ and with taking the large system limit ($K \to \infty,\ \beta \equiv K/N < \infty$), form closed equations containing only macroscopic values. We call the modified algorithm of Tanaka PQΓ-algorithm. With these treatment PQΓ-algorithm for LPIC becomes

$$\boldsymbol{x}^t = a\tilde{\boldsymbol{u}}^{t-1}, \quad \tilde{\boldsymbol{u}}^t = P\left(\boldsymbol{h} - W\boldsymbol{x}^t\right) + (1-P)\tilde{\boldsymbol{u}}^{t-1} - (1-P)P\boldsymbol{x}^{t-1}, \quad P = \frac{1}{1+\beta a}. \tag{2·2}$$

In the linear case the partial cancelation factor P_t is independent of stage number t. The equilibrium of the algorithm is

$$\boldsymbol{x} = a[aW + (1 + a1 - aP)I]^{-1}\boldsymbol{h}. \tag{2·3}$$

Since this should be identical with the result of one shot linear detector, equation (1·4), we obtain the system parameter a as a function of the system load β and receiver-estimated channel noise level σ. However, there are two corresponding system parameters $a_\pm$ for a one shot linear detector as

$$a_\pm = \frac{\beta - 1 - \sigma^2 \pm \sqrt{(\beta - 1 - \sigma)^2 + 4\beta\sigma^2}}{2\beta\sigma^2} \quad \text{for } \sigma > 0 \tag{2·4}$$

$$a_\pm = \begin{cases} \frac{1}{1-\beta}, & -\infty, \quad 0 \le \beta < 1 \\ \infty, & \frac{1}{1-\beta}, \quad 1 \le \beta < \infty \end{cases} \qquad \text{for } \sigma = 0 \tag{2·5}$$

where $a_+ > 0$ and $a_- < 0$. Therefore for a given one shot linear detector there exists two systems of linear PQΓ-algorithm. Stability of equilibrium of linear PQΓ-algorithm is easily evaluated with linear algebra and knowledge of distribution of eigenvalue of random matrix W. The final result is that the algorithm is stable if $\left|\frac{\sqrt{\beta}a}{1+\beta a}\right| < 1$. This condition always holds for the system of $a = a_+$, *i.e.* linear PQΓ-algorithm is convergent for $0 \le \beta < \infty$ and $\sigma \ge 0$, and never holds for the system of $a = a_-$. Thus we can say that the system of $a = a_+$ is completely stable and that of $a = a_-$ is unstable. With the Gaussian assumption for the decision statistics $\tilde{u}_k^t$ and with taking the large system limit we obtain averaged (macroscopic) equations of motion as follows.

$$B_t \equiv \langle \tilde{u}_k^t \rangle = P, \quad C_t \equiv \langle (\tilde{u}_k^t)^2 \rangle = B_t^2[\sigma_0 + \beta(1 - 2M_t + Q_t)]$$

$$M_t = \int Dza(B_t + \sqrt{C_t}z) = aP, \quad Q_t = \int Dza^2(B_t + \sqrt{C_t}z)^2 = a^2(P^2 + C_t)$$

where $Dz \equiv \mathrm{e}^{-z^2/2}/\sqrt{2\pi}$. The bit error rate (BER) is given by $\mathsf{P}_b = Q\left(\frac{B_t}{\sqrt{C_t}}\right)$ with $Q(x) \equiv \int_x^\infty Dz$. The equilibrium BER of linear PQΓ algorithm is obtained as

$$\mathsf{P}_b^{\mathrm{PQ}\Gamma} = Q\left(\frac{B}{\sqrt{C}}\right) = Q\left(\sqrt{\frac{(1+\beta a)^2 - \beta a^2}{\sigma_o^2(1+\beta a)^2 + \beta(1 - a + \beta a)^2}}\right). \tag{2·6}$$

§3. Replica analysis

Tanaka[5] analyzed linear detector with replica method in the scheme of Bayes estimation by assuming the unit-variance Gaussian prior*) $P(\boldsymbol{b}) = (2\pi)^{-K/2}\mathrm{e}^{\|\boldsymbol{b}\|^2}$. The simplified saddle point equations read as follows.

$$m = \frac{E}{1+E}, \quad q = \frac{E^2+F}{(1+E-G)^2}, \quad E = \frac{1}{\sigma^2 + \beta(1-m)}, \quad F = \frac{\sigma_0 + \beta(1-2m+q)}{[\sigma^2 + \beta(1-m)]^2}$$

and the BER is given by

$$\mathsf{P}_b^{\mathrm{RS}} = Q\left(\frac{E}{\sqrt{F}}\right) = Q\left(\sqrt{\frac{(1+E)^2 - \beta)}{\sigma_0^2(1+E)^2 + \beta}}\right). \tag{3·1}$$

From the saddle point equations we obtain two replica solutions:

$$E_\pm = \frac{-\sigma^2 - \beta + 1 \pm \sqrt{(-\sigma^2 - \beta + 1)^2 + 4\sigma^2}}{2\sigma^2} \qquad \text{for } \sigma > 0 \tag{3·2}$$

$$E_\pm = \begin{cases} \infty, & \frac{1}{\beta-1}, \quad 0 \le \beta < 1 \\ \frac{1}{\beta-1}, & -\infty, \quad 1 \le \beta < \infty \end{cases} \qquad \text{for } \sigma = 0, \tag{3·3}$$

*) In the scheme of the Bayes estimation to analyze the one shot linear detector of equation (1·4) corresponds to take Gaussian prior distribution of information bit.[5]

whose stability is examined by $f_{st}(E) \equiv 1 - \beta\frac{E^2}{(1+E)^2}$. Note that in the Gaussian prior case the stability conditions against the perturbation on the replica symmetry plane (the longitudinal mode) and along the replicon mode, the de Almeida-Thouless (AT) condition, are the same and thus RS solution becomes unstable along both directions at the same time. Since the stability of the two solution is examined as $f_{st}(E_+) > 0, \quad f_{st}(E_-) < 0, \quad$ for $\sigma \geq 0$, we know the solution E_+ is stable and E_- is unstable. It can be proved that

$$\mathsf{P}_b^{\mathrm{RS}} = Q\left(\frac{E_\pm}{\sqrt{F(E_\pm)}}\right) = Q\left(\frac{B(a_\pm)}{\sqrt{C(a_\pm)}}\right) = \mathsf{P}_b^{\mathrm{PQ}\Gamma}. \tag{3·4}$$

From this correspondence between RS solution and equilibrium of PQ$\mathit{\Gamma}$ algorithm means that the stable (unstable) RS solution E_+ (E_-) corresponds to stable (unstable) system of PQ$\mathit{\Gamma}$ algorithm a_+ (a_-).

§4. Belief Propagation for the CDMA linear detectors

Following the derivation of the belief propagation (BP) algorithm for CDMA demodulation by Kabashima[6] we derive BP algorithm for linear detector by taking the Gaussian prior distribution of information bit:

$$\boldsymbol{g}^{t+1} = \boldsymbol{h} - W\boldsymbol{m}^t + \beta A^t \boldsymbol{g}^t - \beta A^t \boldsymbol{m}^{t-1} \tag{4·1}$$

where $g_k^t = \sum_{\mu=1}^N \int b_k P^t(y^\mu | b_k, y^{\nu\neq\mu}) Db_k$, $m_k = \int b_k P^t(b_k|, y^\mu) Db_k$, and $A^t = \frac{1}{1+\sigma^2+\beta C^{t-1}}$. C^t is determined by $C^t = \frac{\sigma^2+\beta C^{t-1}}{1+\sigma^2+\beta C^{t-1}}$. If we drop the time dependence of the coefficient A^t and set the parameters as $\boldsymbol{m}^t = \boldsymbol{x}^t$, $\boldsymbol{g}^t = P^{-1}\tilde{\boldsymbol{u}}^{t-1}$, $\beta A^t = 1 - P$ we obtain again the PQ$\mathit{\Gamma}$-algorithm given in equation (2·2). Thus PQ$\mathit{\Gamma}$-algorithm is a special case of BP algorithm.

§5. Discussion

It has been pointed out by Kabashima that the stability of fixed point of the BP algorithm coincide with the AT condition and proved this statement in the marginal-posterior-mode CDMA detection[6] and in a family of spin-glass models.[6] Our results presented here, that stability of PQ$\mathit{\Gamma}$-algorithm (a special case of BP) agrees with stability of the RS solution, support the statement of Kabashima in the CDMA detections with Gaussian prior *i.e.* the linear CDMA detection.

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